Int. J. Solids Structures Vol. 18, No. 7, pp. 551–562, 1982
Printed in Great Britain.

0020–7683/82/070551–12$03.00/0
Pergamon Press Ltd.

STATISTICAL ASPECTS OF THE CONTINUOUS DAMAGE THEORY†

DUSAN KRAJCINOVIC
Department of Materials Engineering, University of Illinois at Chicago Circle, IL 60680, U.S.A.

and

MANUEL AMERICO G. SILVA
Gabinete da Area de Sines, Conselho de Gestao, Lisbon, Portugal

(*Received* 27 *July* 1981; *in revised form* 3 *December* 1981)

Abstract—The present study addresses several fundamental aspects of a link connecting the Continuous Damage Theory and the Statistical Strength Theories. The attention is focused on a simple but efficient way of devising rational damage laws based on the physical nature of the phenomenon. The computed results evidence the remarkable flexibility of the suggested model in its application for a wide range of different materials.

INTRODUCTION

A recently published "philosophic look ahead to the future"[1] is an eloquent testimony on the "renewed emphasis on failure and on response of heavily damaged structures and microstructures." The same author further concludes that "an almost limitless field of useful but difficult research lies ahead in the extension of the damage approach pioneered by Kachanov and Rabotnov."

The essential feature of the Kachanov's model[2] resides in the introduction of a special internal (hidden) variable defining the state of damage locally and recording its accumulation. Leaving aside the disagreement regarding the mathematical nature of the damage variable (discussed at some length by Krajcinovic and Fonseka[3]), the most important and sensitive aspect of a realistic damage model consists of the establishment of a rational damage law (i.e. the response function defining the rate of the damage accumulation in terms of the current values of other state and internal variables). Considering the overwhelming complexity of the physical phenomenon reflecting the rearrangement of the dislocations (plasticity) and the nucleation, growth and coagulation of microdefects (cracks, voids and second phase inclusions—defining the state of the damage) as well as their interaction, the lack of concensus on the most suitable way leading to the establishment of the constitutive equations is not entirely surprising. The existing models can be roughly classified into three different classes:

(i) Purely phenomenological models featuring an *a priori* legislated damage law based on some general speculations and fitting of the existing experimental data[4, 5], etc;

(ii) Theories based on generalizations of the material science models[6, 7];

(iii) Models based on the statistical approach [8,9], etc.

Some of the proposed theories[3] combine the elements of all classes emphasizing that these approaches present but a different view of the same physical phenomenon.

2. FORMULATION OF THE PROBLEM

The present study focuses on the establishment of simple statistical models for the uniaxial response of a gradually damaging structure. The emphasis is placed on the formulation of a simplest possible model allowing the analytical prediction of the general experimentally detected trends of the material behavior. The actual fitting of the experimental data for a specific material is considered beyond the scope of this paper.

A simple physically motivated active damage model (reflecting the mutual interaction of the stress redistribution and damage evolution) for a specimen in uniaxial tension can be established modifying the distributed element model devised by Iwan[10] to study the hysteretic nature of the elasto–plastic response of cyclically loaded tensile specimens.

†This work was supported by the U.S. National Science Foundation grant to the University of Illinois at Chicago Circle.

3. GOVERNING EQUATIONS

3(a) *Perfectly brittle materials*

Consider first the uniaxial tension of a perfectly brittle specimen modeled by a system of parallel bars shown in Fig. 1. Each bar remains completely elastic until it ruptures when the tensile force in the bar reaches its rupture strength $f_i = f_{Ri}$. If the rupture strength is bar dependent, an increment of the external load F acting on the system causes rupture of a certain fraction of bars and the attendant redistribution of forces among the bars which did not rupture.

Assuming that the stiffness of each bar (k/N) is identical and its rupture strength f_{Ri}/N (where N is the original number of bars) different the force-deflection relations are

$$f_i = \frac{kx}{N} \text{ for } 0 \le kx \le f_{Ri} \text{ and } \dot{x} \ge 0$$

$$f_i = 0 \text{ for } kx \ge f_{Ri} \text{ and } \dot{x} \ge 0 \tag{1}$$

where x is the axial displacement of the bar system. Consequently, the equilibrium equation is

$$F = \sum_{n+1}^{N} \frac{kx}{N} \tag{2}$$

where n is the number of the ruptured bars (being a physically acceptable measure of the incurred damage). Thus,

$$F = kx\left(1 - \frac{n}{N}\right) = kx(1-\omega) \tag{3}$$

where the non-negative damage variable ω is defined as a ratio (n/N) of the number of already ruptured bars and the original number of bars. Indeed, since $F = \sigma A_0$, $k = EA_0/L$ and $x = \epsilon L$ where A_0 and L are the original cross-sectional area and the length of the specimen, E its elastic modulus and ϵ the axial strain, the eqn (3) can be rewritten in the familiar form (see Ref. [2])

$$\sigma = E\epsilon(1-\omega) \tag{4}$$

where $\omega = A_v/A_0$ with A_v representing the surface void density.

The actual damage law $\omega = \omega(x)$ is a function of the distribution of the rupture strengths of

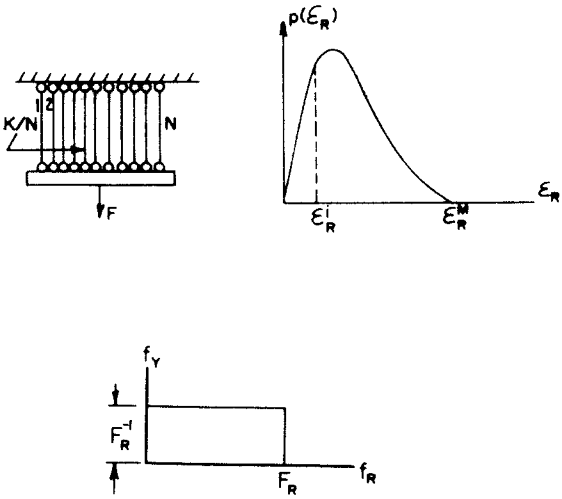

Fig. 1. (a) Parallel bar model. (b) Rupture strength probability density function $p(f_R)$. (c) Uniform rupture strength probability density function.

individual elements. For a very large number of elements N the eqn (2) is recast into an integral form

$$F = kx\int_{kx}^{\infty} p(f_R)\,\mathrm{d}f_R \tag{5}$$

where $p(f_R)$ is the rupture strength probability density function shown in Fig. 1(b). Consequently, $p(f_R)\,\mathrm{d}f_R$ is the fraction of the total number of elements with rupture strengths contained within the interval $[f_R, f_R + \mathrm{d}f_R]$. Since

$$\int_0^{F_R} p(f_R)\,\mathrm{d}f_R = 1 \tag{6}$$

where F_r is the rupture strength of the strongest element, the number of bars which will rupture at a stress level kx is

$$\int_0^{kx} p(f_R)\,\mathrm{d}f_R.$$

Thus the damage is defined as

$$\omega = \frac{\int_0^{kx} p(f_R)\,\mathrm{d}f_R}{\int_0^{F_R} p(f_R)\,\mathrm{d}f_R} = P(kx) = pr(f_R \le kx) \tag{7}$$

where $P(kx)$ is the cumulative distribution function for the given rupture strength probability density function $p(f_R)$. Consequently, the reliability function reflecting the probability that the system will survive at a force level kx is

$$R(kx) = 1 - P(kx) = 1 - \omega. \tag{8}$$

The instantaneous failure rate or the hazard rate, giving the probability that a bar which has survived until the force reached the level kx will fail immediately thereafter (conditional failure rate function) is

$$Z(kx) = \frac{p(kx)}{R(kx)} = \frac{p(kx)}{1-\omega}. \tag{9}$$

The strength distribution commonly identified with the structural failure applications is the Weibull distribution (see, for example, [11]). In case of the Weibull distribution of the rupture strengths the damage law (7) becomes

$$\omega(kx) = 1 - \exp\left[-\left(\frac{kx}{\theta}\right)^m\right] \tag{10}$$

where m and θ are the shape and the scale parameter. The exponential distribution corresponds to the case $m = 1$ and leads to the damage law which for small values of $kx/\theta \ll 1$ reduces to a linear relationship

$$\omega(kx) = \frac{kx}{\theta} = C\epsilon \tag{11}$$

where C is a material constant. The linear damage law (11) has been used rather successfully in the past by Janson and Hult[12], Krajcinovic[13] and Krajcinovic and Forseka[3, 14].

The linear damage law (11) corresponds to a uniform band-limited strength distribution shown in Fig. 1(c). Since from (6) $p(f_R) = 1/F_R$, the expressions for the damage ω and force F are from (11) and (5)

$$\omega = \frac{kx}{F_R} \tag{12}$$

and

$$F = kx(1-\omega) = kx\left(1 - \frac{kx}{F_R}\right). \tag{13}$$

Comparison of (11) and (12) identifies the scale parameter θ as the rupture strength of the strongest member (or the inverse of the average strength distribution $p(f_R)$)

$$\theta = F_R \tag{14}$$

allowing for its unambiguous determination from a simple stress controlled tension test. Defining failure as the inability of a tensile specimen to support additional load increments, from (13)

$$\frac{\partial F}{\partial x} = k\left(1 - 2k\frac{x_f}{F_R}\right) = 0$$

i.e.

$$F_R = 2kx_f \tag{15}$$

where $x = x_f$ is the elongation recorded at the incipient failure. Comparing (12) and (15) it is found that at failure† (in a stress controlled test)

$$\omega_f = 0.5 \text{ and } F_f = 0.25\, F_R. \tag{16}$$

For the selected strength distribution from (8) and (9)

$$R(kx) = 1 - \frac{kx}{F_R} \qquad Z(kx) = \frac{1}{(1-\omega)F_R} \tag{17}$$

such that

$$R(kx_f) = 0.5 \text{ and } F_R Z(kx_f) = 2. \tag{18}$$

In a strain controlled experiment, the softening part of the force-deflection curve can be obtained (Fig. 2). The failure occurs at $\omega = 1$ as hypothesized originally by Kachanov[2] and Rabotnov[15].

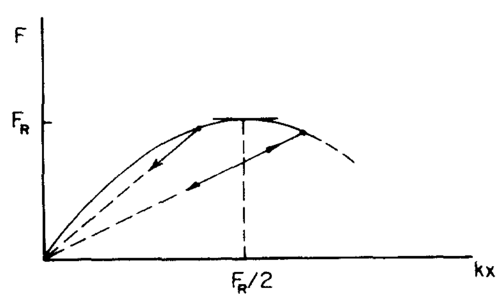

Fig. 2. Load-displacement curve for a brittle specimen (with linear damage law) in tension.

†Here the word failure implies in essence instability.

Since the cleavage rupture mode of brittle materials is physically identified with tension, it appears reasonable to postulate that

$$d\omega = 0 \quad \text{when} \quad \dot{x} < 0. \tag{19}$$

Consequently in unloading the force-displacement relation is defined instead of (13) by

$$F = kx(1-\bar{\omega}) = \bar{k}x \tag{20}$$

where

$$\bar{k} = k(1-\bar{\omega}) \tag{21}$$

with $\bar{\omega}$ being the largest previously accumulated (recorded) level of damage. It is interesting to note that the unloading curve is a straight line pointed towards the origin ($F = 0$, $x = 0$) as shown in Fig. 2. It is also important to point out that no residual plastic deformation exists once the specimen is fully unloaded and that the stiffness of the specimen is degraded $\bar{k} < k$ (21) as a result of the accumulated damage. This phenomenon of gradually diminishing values for stiffness is especially noticeable in the case of brittle rocks (see[16]).

It is informative to examine the strain energy of the system. The strain energy can be written as

$$U = \tfrac{1}{2}Fx = \tfrac{1}{2}kx^2(1-\omega). \tag{22}$$

The thermodynamic force G conjugate to the kinematic damage variable ω is

$$G = -\frac{\partial U}{\partial \omega} = \tfrac{1}{2}kx^2. \tag{23}$$

As noticed by Chaboche[17] and Krajcinovic and Fonseka[3], the thermodynamic force G is actually the crack resistance force (or the energy release rate). The dissipation power density is, in absence of the plastic flow, given simply by

$$dD = G\,d\omega = \tfrac{1}{2}kx^2\,d\omega \tag{24}$$

which in conjunction with (12) leads to

$$\frac{dD}{d\omega} = 2F_R^2\left(1 - \sqrt{\left(1 - 4\frac{F}{F_R}\right)}\right)^2. \tag{25}$$

At $F = \frac{1}{4}F_R$, it follows from (25) that $dD/d\omega = 0$. Since the energy is still being dissipated, it follows that at the rupture $F \to \frac{1}{4}F_R$; the damage accumulation rate $d\omega$ becomes infinite or conversely that the crack resistance force G vanishes. It is also important to notice that the local entropy production is always non-negative; i.e. $dD \geq 0$ as a consequence of (24) and $d\omega \geq 0$ (in absence of healing). Thus the selected model satisfies the governing thermodynamic principles (Clausius–Duhem inequality).

The force-displacement relationship (13), in case of a uniform band-limited probability density function $p(f_R)$ is a second order parabola (Fig. 2) symmetric about the axis $F = F_{max}$. While such a parabola fits surprisingly well the experimental data for the high strength concrete[13], a set of different and possibly more accurate damage laws can be derived on the basis of other strength probability distributions. Nevertheless, it is difficult to disregard this simple model based on a single additional, physically identifiable, material parameter.

3(b). *Brittle–ductile materials*

Even though its ductility may be inhibited by low temperatures (below the transition point), high strain rates or various chemical changes, the commonly used materials (with a possible exception of some ceramics, glass and natural materials such as rock) exhibit certain levels of capability to deform plastically. Thus, in general, the energy imparted by a given increment of the external load is:

(i) Partially stored as an elastic strain energy; and

(ii) Partially dissipated in order to rearrange the dislocation pattern and increase the level of the microdefect density.

Consequently, in addition to the state variable defining the reversible process (say, elastic strain), it is necessary to introduce two independent sets of internal variables defining the two dissipative processes. The plastic (ductile) behavior is typically defined by the hardening variables related to the plastic strain. The state of the microcracking (often manifested as softening) locally is again conveniently defined by the internal variable ω which serves as a repository of the recorded history related to the accumulated damage. The mutual interdependence of these two processes is obviously of fundamental importance in establishment of a rational analytical model.

The actual distinction between these two physically different processes can be readily discerned from a simple tensile experiment (Fig. 3). In case of a perfectly brittle material (Fig. 3a) the deformation completely vanishes upon unloading since the unruptured material remains perfectly elastic. In case of a perfectly ductile material (Fig. 3b) the unloading part of the stress-strain curve is (at least initially) parallel to the initial part of the loading curve (E_{el} = const.). If the response of the material is governed by both processes (Fig. 3c), the unloading curve will be somewhat less sloped ($E_0(1-\bar{\omega}) < E_0$) than the initial part of the loading curve (E_0). The existence of the residual strain x_p is, on the other hand, indicative of the plastic deformation.

The major problem is establishment of a rational phenomenological analytical model combining these two dissipative processes is again related to the definition of a reasonably simple response function (constitutive equation or damage law) modeling adequately the most salient features of the material response (such as the existence of the plastic hysteresis, Bauschinger effect, etc.). It is, naturally, again possible to fit a certain number of experimentally obtained curves in hope that the obtained relations will subsequently approximate the behavior of the material under different circumstances (see, for example, [17]). It is somewhat more appealing to examine a physically motivated geometrically simple model (such as the one used in the previous section of this paper) in order to determine the physical basis of the relations between the governing state and internal variables.

The geometrically simplest model consists of a parallel connection of Jenkin's elements (Fig. 4) combining the ductile model used by Iwan[10] and the brittle model used previously in this paper. Each element is attributed a random value of the yield f_{yi} and rupture strength f_{Ri}. In absence of hardening the element "i" will either yield $f_{Ri} > f_{yi}$ or fracture $f_{Ri} < f_{yi}$, i.e. the events are mutually exclusive. The degree of ductility is controlled through the selection of the joint probability density function $p(f_y, f_R)$, (Fig. 5).

It is often convenient to introduce the brittleness index representing the fraction of the brittle bars, i.e. the ratio of areas A_b and $A = A_b + A_d$ (Fig. 5b). Thus the system for which $b = 1$ is perfectly brittle, while $b = 0$ implies a perfectly ductile system.

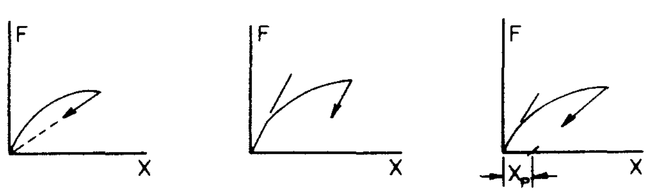

Fig. 3.

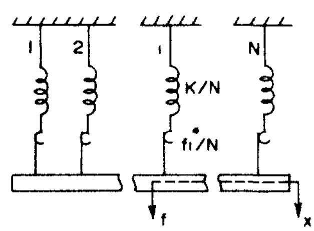

Fig. 4.

Fig. 3. (a) Tension experiment on a perfectly brittle specimen. (b) Tension experiment on a perfectly ductile specimen. (c) Tension experiment on a brittle–ductile specimen.

Fig. 4. Parallel connection of Jenkin's elements.

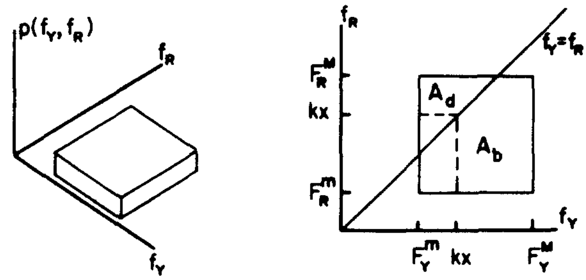

Fig. 5. (a) Uniform joint probability density function. (b) Projection of the uniform probability density function on (f_R, f_y) plane.

In addition, the fraction of bars remaining elastic at a force level kx/N is given by the complement of the cumulative distribution function $[1 - P(kx, kx)]$.

In order to illustrate the application of the proposed analytical model and avoid unnecessary complications, consider the simplest case of a uniform joint probability density function $p(f_y, f_R) = [(F_y^M - F_y^m)(F_R^M - F_R^m)]^{-1} = \text{const.}$ shown in Fig. 5. The force-deflection relations are:

$$\begin{aligned} f_i &= 0 && \text{in Region 1} && (kx > f_{Ri},\ f_{Ri} < f_{yi}) \\ f_i &= f_{yi}/N && \text{in Region 2} && (kx > f_{yi},\ f_{Ri} > f_{yi}) \\ f_i &= kx/N && \text{in Region 3} && (kx < \min(f_{yi}, f_{Ri})). \end{aligned} \tag{26}$$

For the present purposes assume also that for a specific material

$$0 \le F_R^m \le F_y^m \le F_R^M \tag{27}$$

which merely implies that the response of the material at very low load levels is governed by some initial microdefect density (prior to the onset of yielding).

The brittleness index defined as the fraction of the brittle bars is

$$b = \int_{F_y^m}^{F_y^M} \mathrm{d}f_y \int_{F_R^m}^{f_y} p(f_y, f_R)\,\mathrm{d}f_R = \tfrac{1}{2}\frac{F_y^M + F_y^m - 2F_R^m}{F_R^M - F_R^m}; \quad \text{for } F_y^M \le F_R^M$$

or (28)

$$b = 1 - \int_{F_y^m}^{F_R^M} \mathrm{d}f_R \int_{F_y^m}^{f_R} p(f_y, f_R)\,\mathrm{d}f_y = 1 - \tfrac{1}{2}\frac{(F_R^M - F_y^m)^2}{(F_y^M - F_y^m)(F_R^M - F_R^m)}; \quad \text{for} \quad F_y^M \ge F_R^M.$$

Response of the system can be divided into three different phases:

(i) *Elastic phase*

None of the members yields or ruptures. Hence

$$\left.\begin{aligned} F &= kx \\ \omega &= 0 \end{aligned}\right\} \quad \text{for} \quad \dot{x} \gtreqless 0 \quad kx \le \min(F_R^m, F_y^m). \tag{29}$$

Unloading follows the same path.

(ii) *Brittle phase* $(f_r^m \le kx \ge F_y^m)$

The bars for which $f_{Ri} < kx$ will rupture. At the same time since $kx \leqslant F_y^m$ no bar will yield. Hence the incurred damage is again equal to the cumulative distribution function

$$\omega = P(kx) = \int_{F_R^m}^{kx} \mathrm{d}f_R \int_{F_y^m}^{F_y^M} p(f_y, f_R)\,\mathrm{d}f_y = \frac{kx - F_R^m}{F_R^M - F_R^m}. \tag{30}$$

The equilibrium equation is simply

$$F = kx\int_{kx}^{F_R^M} df_R \int_{F_y^m}^{F_y^M} p(f_y, f_R)\,df_y = kx[1 - P(kx)] = kx(1-\omega) \tag{31}$$

where ω is given by (30).

In unloading the load-deflection curve is a straight line defined by

$$F = kx(1-\bar{\omega}) \quad \text{for} \quad \dot{x} < 0 \tag{32}$$

where $\bar{\omega}$ is determined for the maximum deflection x (which is still below F_y^m/k) from (30).

(iii) *Brittle–ductile phase* $F_y^m \le kx \le \min(F_y^M, F_R^M)$

The bars for which $f_{Ri} < kx$, $f_{Ri} < f_{yi}$ will rupture while the bars for which $f_{yi} < kx$, $f_{yi} < f_{Ri}$ will yield. The equilibrium equation is

$$F = F_E + F_p \tag{33}$$

The resultant of forces in elastic bars is

$$F_E = kx\int_{kx}^{F_R^M} df_R \int_{kx}^{F_y^M} p(f_y, f_R)\,df_y = \frac{kx}{P}(F_y^M - kx)(F_R^M - kx) \tag{34}$$

where $P = (F_y^M - F_y^m)(F_R^M - F_R^m)$. The total force accumulated in yielded bars is

$$F_p = \int_{F_y^m}^{kx} df_R \int_{F_y^m}^{f_R} f_y p(f_y, f_R)\,df_y + \int_{kx}^{F_R^M} df_R \int_{F_y^m}^{kx} f_y p(f_y, f_R)\,df_y$$

$$= \frac{1}{2P}\left[-\frac{2}{3}(kx)^3 + F_R^M(kx)^2 + \frac{2}{3}(F_y^m)^3 - (F_y^m)^2 F_R^M\right]. \tag{35}$$

Here the total force is from (33 to 35) during the loading ($\dot{x} > 0$)

$$F = \frac{1}{P}\left[\frac{2}{3}(kx)^3 - \frac{1}{2}(kx)^2(2F_y^M + F_R^M) + kxF_R^M F_y^M + \frac{1}{3}(F_y^m)^3 - \tfrac{1}{2}F_R^M(F_y^m)^2\right] \tag{36}$$

if $F_y^m \le kx \le \min(F_y^M, F_R^M)$. The accumulated damage (i.e. the fraction of ruptured members) is

$$\omega = \frac{1}{P}[F_y^m(F_R^m - \tfrac{1}{2}F_y^m) - F_R^m F_y^M + kxF_y^M - \tfrac{1}{2}(kx)^2]. \tag{37}$$

For unloading ($\dot{x} < 0$) it is again reasonable to assume that the damage remains constant $\omega = \bar{\omega}$, $\dot{\omega} = 0$ where $\bar{\omega}$ is computed from (37) for maximum x (at which $\dot{x}$ reverses sign). In addition, it becomes necessary to define the yield condition in the domain of the compressive stresses. Regardless of the selected yield law, the unloading commences along a straight line

$$F = kx(1-\bar{\omega}). \tag{38}$$

Thus since the total deflection can be written as a sum of the elastic and plastic component

$$x = x_E + x_P \tag{39}$$

it follows that the plastic deformation accumulated at an arbitrary stage of the loading process is

$$kx_p = kx - \frac{F}{1-\bar{\omega}} \tag{40}$$

where kx and $\bar{\omega}$ are computed from (36) and (37).

The process terminates with the force F reaching the $\min(F_y^M, F_R^M)$. At this point all bars have either yielded or ruptured and the system cannot accommodate further load increase. The failure of the system is either

(i) Ductile if $F_y^M < F_R^M$ (horizontal tangent on the load-displacement curve at x_R) or;
(ii) Brittle if $F_y^M > F_R^M$ (negative slope of the tangent).

4. ILLUSTRATIVE EXAMPLES

4(a) *Brittle material*

Consider first the brittle materials with a Weibull distribution of the rupture strength. The influence of the shape factor m on the force-displacement curve is illustrated by graphs in Fig. 6. These curves are remarkably similar to curves characterizing the behavior of concrete. For greater generality both the load and the displacement are divided by the shape factor θ. The two parameters m and θ can be determined from the maximum load

$$F_{\max} = \theta(me)^{-1/m} \tag{41}$$

and an additional measurement at the post-failure part of the curve. An additional possibility would involve the determination of ω at $F_{\max}$. From (3) and (10)

$$\omega_m = 1 - \exp\left[-\left(\frac{kx_f}{\theta}\right)^m\right] \tag{42}$$

where $\omega = \omega_m$, $F = F_{\max}$ at $x = x_f$. Two equations (41) and (42) suffice for the determination of the two parameters m and θ. It is important to notice that the void density at incipient failure was successfully measured in the past by Curran *et al.*[18] for a polycarbonate plate, Shockey, *et al.*[19] for α-titanium, McHugh *et al.*[20] for rock shale, etc. It is quite plausible that ω_m $(T, \dot{\epsilon})$ is a material constant (dependent on temperature T and strain rate $\dot{\epsilon}$. A simpler, if not more accurate, way to determine ω is to measure the slope of the unloading curve and use eqn (20).

4(b) *Brittle–ductile material*

Introducing the nondimensional quantities

$$\xi = \frac{kx}{F_y^M} \quad \bar{f} = \frac{F}{F_y^m} \quad \alpha = \frac{F_y^M}{F_R^M}$$

$$\beta_y = \frac{F_y^m}{F_y^M} \quad \beta_R = \frac{F_R^m}{F_R^M} \quad \xi_p = \frac{kx_p}{F_y^M}. \tag{43}$$

The brittleness index (28) becomes

$$b = \frac{\alpha}{2}\,\frac{1 + \beta_y - (2\beta_R/\alpha)}{1 - \beta_R} \quad \text{for} \quad \alpha \le 1$$

and (28′)

$$b = 1 - \frac{(1 - \alpha\beta_y)^2}{2\alpha(1 - \beta_R)(1 - \beta_y)} \quad \text{for} \quad \alpha \ge 1.$$

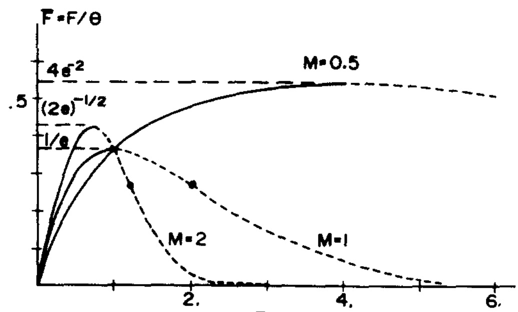

Fig. 6. Force-displacement curves for three different values of the shape factor m in Weibull's distribution of rupture strengths.

The governing equations (29–36) can now be recast into a form more suitable for computations.

(i) *Elastic phase* ($\xi \le 1/\alpha$)

$$\bar{f} = \xi \quad \text{for} \quad \dot{\xi} \gtreqless 0 \quad \text{and} \quad \xi \le \beta_R/\alpha \tag{29'}$$

and $\omega = 0$.

(ii) ***Brittle phase*** ($\beta_R/\alpha \le \xi \le \beta_y$)

$$\left.\begin{aligned} \bar{f} &= \xi(1-\omega) \\ \omega &= (\alpha\xi - \beta_R)(1-\beta_R) \end{aligned}\right\} \quad \text{for} \quad \dot{\xi} > 0 \qquad \begin{aligned}(31')\\(30')\end{aligned}$$

and

$$\bar{f} = \xi(1-\bar{\omega}) \quad \text{for} \quad \dot{\xi} < 0 \tag{32'}$$

(iii) ***Brittle–ductile phase*** ($\beta_y \le \xi \le \min(1, 1/\alpha)$)

$$\bar{f} = \frac{\alpha}{(1-\beta_y)(1-\beta_R)}\left(\frac{2}{3}\xi^3 - \frac{1+2\alpha}{2\alpha}\xi^2 + \frac{1}{\alpha}\xi + \frac{1}{3}\beta_y^3 - \frac{1}{2\alpha}\beta_y^2\right) \tag{36'}$$

$$\omega = \frac{\beta_y(\beta_R - \frac{1}{2}\alpha\beta_y) - \beta_R + \alpha\xi(1-\frac{1}{2}\xi)}{(1-\beta_y)(1-\beta_R)} \tag{37'}$$

$$\xi_p = \xi - \frac{\bar{f}}{1-\omega} \tag{40'}$$

for $\dot{\xi} > 0$. The condition (27)

$$0 \le \frac{1}{\alpha}\beta_R \le \beta_y \le \frac{1}{\alpha}. \tag{27'}$$

The numerical results presented in the sequel are generated for $\beta_R = 0$ (implying existence of an initial microdefect distribution) and $\beta_y = 0.2$. The brittleness index was (arbitrarily) selected as

(a) $\alpha = 0.5$ for which $b_3 = 0.3$ ("ductile")
(b) $\alpha = 1.0$ for which $b_2 = 0.6$
(c) $\alpha = 2.0$ for which $b_1 = 0.89$ ("brittle").

The force-deflection curves for these three hypothetical materials are plotted in Fig. 7. These curves bear remarkable similarity to curves measured in strain-controlled test (see[14]). The

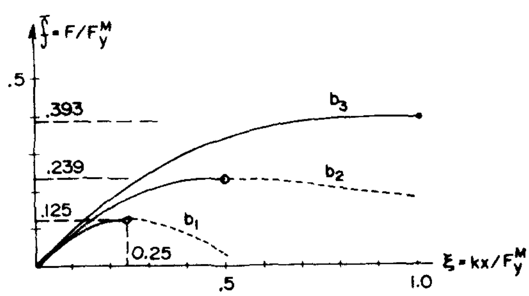

Fig. 7. Force-displacement curves for three materials with different brittleness indices ($b_3 = 0.30$, $b_2 = 0.60$, $b_1 = 0.89$).

maximum force occurs at $\xi_f = 1/2\alpha$. For the same k and F_y^M the maximum force is reached much earlier in case of brittle materials. As expected, ξ_f also decreases as the brittleness increases. The values of ω at $x = x_f$ were computed to be $\omega = 0.393, 0.239, 0.125$ for $b = 0.30, 0.60, 0.89$.

The damage-displacement curves are plotted in Fig. 8. In case of brittle materials the damage is virtually a linear function of displacement x. For ductile materials the later stages are dominated by the plastic deformation at a modest rate of damage growth.

The plastic strain-total strain curves are plotted in Fig. 9. In case of brittle materials the growth of plastic strains is confined almost exclusively to the incipient failure phase. The maximum plastic strains were $\xi_p = 0.438, 0.536$ and 0.200 for $b = 0.3, 0.6, 0.89$, respectively.

Naturally, the number of material parameters is in this case larger (F_y^M, F_y^m, F_R^M, F_R^m in addition to k). Proper determination of these constants would require several different tests involving both loading and unloading. Using the same parallel bar model and assuming that rupturing occurs only in tension, it is rather straightforward to obtain the curve for first $1\frac{1}{2}$ load cycles shown in Fig. 10 (computed for $\beta_R = \beta_y = 0$ and $\alpha = 1$, i.e. $b = 0.5$). The hysteretic nature of the curve is quite apparent. One notes also that if F_y^M and F_R^M are constants a plastic-rupture shakedown develops. Hence in order to be able to model the fatigue it is necessary to develop a proper functional relationship between the involved material parameters.

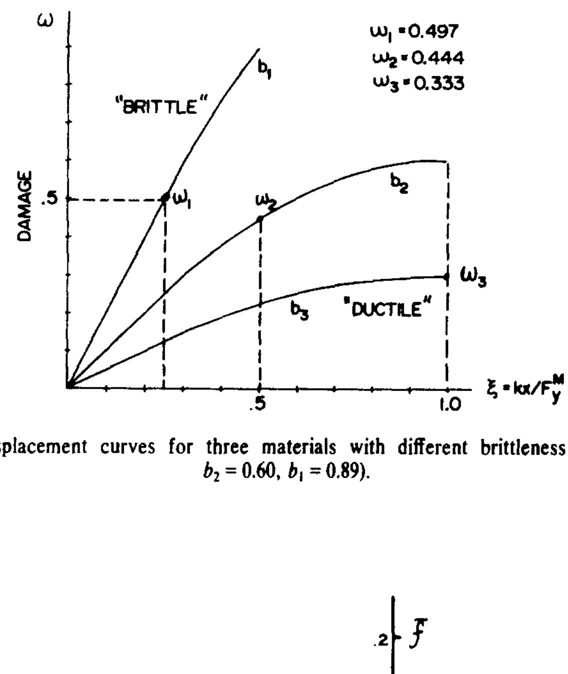

Fig. 8. Damage-displacement curves for three materials with different brittleness indices ($b_3 = 0.30$, $b_2 = 0.60$, $b_1 = 0.89$).

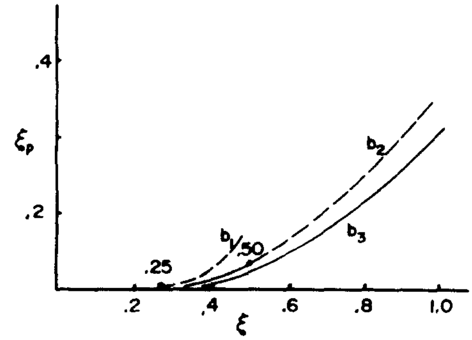

Fig. 9.

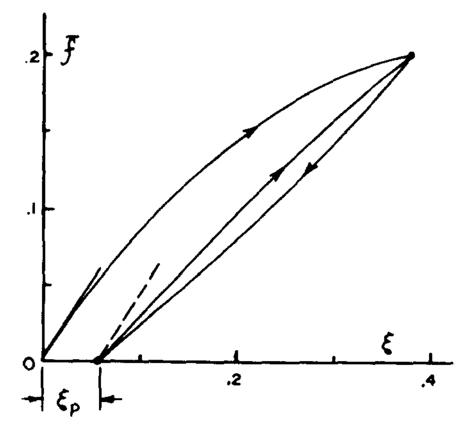

Fig. 10.

Fig. 9. The plastic vs total displacement curves for three materials with different brittleness indices.

Fig. 10. Force-displacement curves for $1\frac{1}{2}$ load cycle (brittleness ratio $b = 0.5$).

SUMMARY AND CONCLUSIONS

The present study explores some of the elementary statistical aspects of the continuous damage theory. The emphasis is placed on the

(i) Physical clarity of the model;

(ii) Wide range of significantly different load-deflection curves which can be generated by this simple model;

(iii) A decidedly modest number of the introduced material parameters sufficient for the description of a complex physical phenomenon; and

(iv) Physical identication of these material parameters.

It would be pretentious to conclude that the presented model is the only, or even the best, way leading to the resolution of the complex problem of the gradual degradation of materials exposed to large tensile stresses. Nevertheless, in its simplicity it still contains most of the salient features of the phenomenon and hints at an intriguing link to the statistical strength theories.

REFERENCES

1. D. C. Drucker, Some classes of inelastic materials related problems basic to future technologies. *Nucl. Engng Design* **57**, 309–322 (1980).
2. L. M. Kachanov, On the creep fracture time. *Izv. AN SSSR, OTN* **8**, 26–31 (1958). (In Russian.)
3. D. Krajcinovic and G. U. Fonseka, The continuous damage theory of brittle materials—I. General theory. *J. Appl. Mech.* **48**, 809–815 (1981).
4. L. Davison and A. L. Stevens, *Thermomechanical constitution of spalling elastic bodies. J. Appl. Phys.* **44**, 667–674 (1973).
5. J. Lemaitre and J. L. Chaboche, Aspect phenomenologique de la rupture par endommagement *J. Mec. Appl.* **2**, 317-365 ((1978).
6. F. A. Leckie and D. R. Hayhurst, Creep rupture of structures. *Proc. Roy. Soc. London A* **340**, 323–347 (1974).
7. L. Seaman, D. R. Curran and D. A. Shockey, Computational models for ductile and brittle fracture. *J. Appl. Phys.* **47**, 4814–4826 (1976).
8. A. I. Chudnovski, A theory of long time strength in fatigue and creep. Workshop on Cont. Mech. Approach to Damage and Life Prediction, pp. 79–87, Gen. Butler St. Park, KY, 1980.
9. H. Mihashi and F. H. Wittmann, Stochastic approach to study the influence of rate of loading on strength of concrete. Heron **25**, 5–54 (1980).
10. W. D. Iwan, On a class of models for the yielding behaviour of continuous and composite systems. *J. Appl. Mech.* **34**, 612–617 (1967).
11. J. B. Kennedy and A. M. Neville, *Basic Statistical Methods for Engineers and Scientists*, 2nd Ed. IEP, New York (1964).
12. J. Janson and J. Hult, Fracture mechanics and damage mechanics, a combined approach. *J. Mec. Appl.* **1**, 69–84 (1977).
13. D. Krajcinovic, A distributed damage theory of beams in pure bending. *J. Appl. Mech.* **46**, 592–596 (1979).
14. G. U. Fonseka and D. Krajcinovic, The continuous damage theory of brittle materials—II. Uniaxial and plane response modes. *J. Appl. Mech.* **48**, 816–824 (1981).
15. Yu. N. Rabotnov, On the equations of state for creep *Progress in Appl. Mech.* pp. 307–315. The Prager Anniversary Volume (1963).
16. W. R. Wawersik, and C. Fairhurst, A study of brittle rock fracture in laboratory compression experiments *Int. J. Rock Mech. Min. Sci.* 7, 561–575 (1970).
17. J. L. Chaboche, Description thermodynamique et phenomenologique de la viscoplastic cyclique avec endommagement. ONERA, Publ. No. 1978-3, 1978.
18. D. R. Curran, D. A. Shockey and L. Seaman, Dynamic fracture criteria for a polycarbonate. *J. Appl. Phys.* **44**, 4025–4038 (1973).
19. D. A. Shockey, K. C. Dao and R. L. Jones, Effect of grain size on the static and dynamic fracture behavior of α-titanium. SRI International Report (undated).
20. S. L. McHugh, D. R. Curran and L. Seaman, The nag-frag computational fracture model and its use for simulating fragmentation and fracture. Presented at the 1980 SESA metting, Ft. Lauderdale, Oct. 1980.